Pullbacks of ∞-Topoi; or, How I Learned to Stop Worrying and Love Gabriel-Ulmer Duality

§1 Stratified Topoi

Want to study stratified geometric objs, e.g. topological spaces, manifolds, schemes.

Stratifying points is one approach, but doesn't always work well. Need to worry about point-set nonsense.

Other option: stratify the sheaf topos!

A given subset of a scheme may or may not yield a subscheme, but a subterminal obj of $Shv(X)$ always will.

Doing derived/spectral AG $\Rightarrow$ need ∞-topoi, not 1-topoi. This has been studied by ([Barwick-Glasman-Haine], Exodromy).

What actually is a stratification?

<u>Idea</u>: Filter X over poset P. "Strata" are the fibers of elements.

<u>Ex</u>: $(\mathcal{M}_{fg})_p$ has ht filtration over $\mathbb{N}$.

The following def is due to BGH.

<u>Def</u>: A stratification of $X \in \mathcal{R}Top_\infty$ over a poset P is a geometric morphism

$$\begin{tikzcd} X \arrow[d] \\ Shv(Open(P)) \end{tikzcd}$$

Here, P has the <u>Alexandrov topology</u>: $U \subset P$ is open iff it is upwards-closed.

Note that the Alexandrov space of P is canonically w.h.e. to the nerve of P thought of as a category, so this is a sensible definition.

<u>Thm</u>: P finite $\Rightarrow Shv(Open(P)) \simeq Fun(P, \mathcal{S})$.

This is further justification. BGH restrict themselves to finite posets, but I don't.

<u>Def</u>: For $p \in P$, the pth stratum is the pullback

$$\begin{tikzcd} X_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ Shv(*) \arrow[r, "p"'] & Shv(Open(P)) \end{tikzcd}$$

That's enough motivation, I think.

§2 Lurie's Recipe

HTT Prp 6.3.4.6 tells us that $\mathcal{R}Top$ has pullbacks. The proof also tells us how to compute them... kind of.

Suppose our cospan diagram looks like this,

$$\begin{tikzcd} & \mathcal{P}(\mathcal{D}') \arrow[d, "\mathcal{P}(f)"] \\ \mathcal{P}(\mathcal{D}) \arrow[r, "\mathcal{P}(g)"] & \mathcal{P}(\mathcal{C}) \end{tikzcd}$$

where

- $\mathcal{P}$ is the contravariant presheaf functor
- $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{D}'$ have finite limits, and
- f and g preserve finite limits.

Then we can take the pushout

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "g"'] & \mathcal{D}' \arrow[d] \\ \mathcal{D} \arrow[r] & \mathcal{E} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

in Cat^{lex}, and

$$\begin{tikzcd} \mathcal{P}(\mathcal{E}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{P}(\mathcal{D}') \arrow[d] \\ \mathcal{P}(\mathcal{D}) \arrow[r] & \mathcal{P}(\mathcal{C}) \end{tikzcd}$$

will be a pullback in $\mathcal{R}Top$.

Of course, not every topos is a presheaf topos. <u>However</u>:

- If
$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$
is a pb, and Z is a left-exact localization of Z', then
$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ Y \arrow[r] & Z' \end{tikzcd}$$
is also a pb.

- If
$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$
is a pb, then for any l.e. loc's $S^{-1}X$ and $T^{-1}Y$,
$$\begin{tikzcd} U^{-1}W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & S^{-1}X \arrow[d] \\ T^{-1}Y \arrow[r] & Z \end{tikzcd}$$
is also a pb, where U is generated by the images of S, T under the left adjoints.

As it turns out (details in HTT), you can always use these to reduce to the special case.

Actually doing so requires either some guesswork or some computations w/ universal properties. In my case, I was lucky and the correct functors were relatively obvious.

So we've reduced to computing a pushout in Cat^{lex}. How do we do that?

§3 Gabriel-Ulmer Duality

There is a beautiful thm (HTT Prp 5.5.7.8) saying that we have an equivalence

$$\begin{tikzcd} Cat^{rex}_{idem} \arrow[r, bend left, "Ind"] \arrow[r, phantom, "\sim"] & Pr^L_\omega \arrow[l, bend left, "(-)^\omega"] \end{tikzcd}.$$

<u>Big diagram</u>:

$$\begin{tikzcd} Cat^{lex} \arrow[r, bend right, "T"'] & Cat^{lex}_{idem} \arrow[l, bend right] \\ \wr op & \wr op \\ Cat^{rex} \arrow[r, bend right, "T"'] & Cat^{rex}_{idem} \arrow[l, bend right] \arrow[d, bend left, "Ind"] \arrow[d, phantom, "\wr"] \\ & Pr^L_\omega \arrow[u, bend left, "(-)^\omega"] \underset{\substack{right \\ adjoint}}{\simeq} (Pr^R_\omega)^{op} \longrightarrow \widehat{Cat}^{op} \end{tikzcd}$$

The inclusion $(Pr^R_\omega)^{op} \to \widehat{Cat}^{op}$ preserves colimits, so we can compute pushouts in Cat^{lex}_{idem} by going all the way through, computing a pullback of categories, and then going all the way back.

Note that idempotent-completion is really just adjoining a filtered colimit, so it doesn't change $\mathcal{P}(\mathcal{C})$ and plays nice with the big diagram. Therefore, no need to assume our cats are idempotent-cplt if we only care about the topos.

§4 My Example

Want to compute pb of

$$\begin{tikzcd} & Shv^{pro\acute{e}t}(Isog) \arrow[d, "ram_1"] \\ \mathcal{S} \arrow[r, "(r)"'] & Shv(Open(Prim(R)^{op})) \end{tikzcd}$$

Turns out this is $\mathcal{P}$(a diagram in Cat^{lex}):

$$\begin{tikzcd} & {[1]} \\ Isog_{pro\acute{e}t} & Open(Prim(R)^{op}) \arrow[l, "G"'] \arrow[u, "\delta_{(r)}"'] \end{tikzcd}$$

$$G(U) = \underset{s \in U^{op}}{colim}\, Isog^{\geq s}.$$

Take Pro of this. Note that for a poset P, pro-objects corr to filters as in combinatorics.

(A <u>filter</u> of a poset is a nonempty subset which is upwards-closed and downwards-directed.)

So we want to compute the pullback in $\widehat{Cat}$ of

$$\begin{tikzcd} & {[1]} \arrow[d] \\ Pro(Isog_{pro\acute{e}t}) \arrow[r] & Filt(Open(Prim(R)^{op}))^{op} \end{tikzcd}.$$

This done easily enough. The pullback σ_r is the full subcategory of $pro(Isog_{pro\acute{e}t})$ on

i) Pro-stacks which admit a map to $Isog^{\geq r}$ but not $Isog^{\geq s}$ for r strictly dividing s, and

ii) The "empty stack" $\emptyset$.

These correspond to 1 and 0 respectively.

Unsurprisingly, the cocompact objects in σ_r are just the representable pro-objects. So we have

$$\begin{tikzcd} S_r \arrow[r, shift left] \arrow[d, shift right] & {[1]} \arrow[l, shift left] \arrow[d, shift left] \\ Isog_{pro\acute{e}t} \arrow[u, shift right] \arrow[r, shift right] & Open(Prim(R)^{op}) \arrow[l, shift right] \arrow[u, shift left] \end{tikzcd}.$$

The localization of $\mathcal{P}(S_r)$ is just the sheaf category of the induced topology, namely the fpqc topology.